Valuation of options on discretely sampled variance: A general analytic approximation*

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Abstract

The values of options on realized variance are significantly impacted by the discrete sampling of realized variance and may be substantially higher than the values of options on continuously sampled variance. Under general stochastic volatility dynamics, we analyze the discretization effect and obtain an analytical correction term to be applied to the value of options on continuously sampled variance. The result allows for a straightforward implementation in many of the standard stochastic volatility models proposed in the literature. Finally, we compare the performance of different numerical methods for pricing options on discretely sampled variance and give recommendations based on the option's characteristics.

KEYWORDS: options on realized variance, variance swaps, stochastic volatility.
JEL: G13, C63.

1 Introduction

Early literature on variance derivatives assumed continuous sampling of the realized variance. The first contribution belongs to Dupire(1993) and Neuberger(1994) who introduced the concept of the log-contract and argued that delta hedging this contract leads to the replication of the continuously

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sampled variance. Carr, Madan(1998) subsequently extended the results and showed how variance swaps (as well as corridor variance swaps) can be replicated by a static position in a continuum of vanilla options, dynamically delta-hedged with the underlying asset. Broadie, Jain(2008) further show that volatility derivatives can be dynamically hedged using variance swaps and a finite number of European options. For a detailed overview of the various theoretical developments, we refer the reader to Carr, Lee(2009).

In derivative markets, variance contracts are specified with discrete sampling; in particular, daily sampling is the most common convention. For linear contracts on realized variance (such as variance swaps), the discretization effect is usually small, as found in Broadie, Jain(2008). The explanation follows from the fact that linear contracts on variance do not depend on the volatility of variance. Itkin, Carr(2010) used the method of forward characteristic functions to show how to price discrete variance swaps in general time-changed Lévy models. Under the assumption of a stochastic clock independent of the driving Lévy process, section 6 in Carr, Lee and Wu(2010) shows that discrete sampling increases the value of variance swaps.

For non-linear contracts on variance (such as options on variance) the discretization effect becomes substantial, especially for shorter maturities. The short-time limit of the discretization gap, under general semi-martingale dynamics, has been derived recently in Keller-Ressel, Muhle-Karbe(2011); the authors also develop Fourier pricing methods for options on discrete variance under exponential Lévy dynamics. In the context of the Heston(1993) model, Sepp(2012) proposes an approximation by combining the distribution of quadratic variation in the Heston(1993) model with that of discrete variance in an independent Black, Scholes(1973) model. The approach leads to a tractable characteristic function for the discretely sampled variance and provides good accuracy near the at-the-money region, across maturities.

In this paper we provide a comprehensive treatment of the discretization effect under general stochastic volatility dynamics. Additionally, we do not restrict attention to particular strike ranges or to particular maturity ranges. We begin by proving that, conditional on the realization of the instantaneous variance process, the (properly scaled) residual randomness arising from discrete sampling can be well approximated with a normally distributed random variable (as formulated in Theorem 2.1 of the following section). In the financial econometrics literature, see Barndorff-Nielsen, Shephard(2002), and more generally, Barndorff-Nielsen et al.(2006), related results were obtained and used in the analysis of high frequency data and the estimation of stochastic volatility models. In contrast, we adopt a conditional approach, which was pioneered in the study of stochastic volatility models by Hull, White(1987), in the case of zero correlation between volatility and the underlying asset, and by Romano, Touzi(1997) and Willard(1997) in the case of non-zero correlation. Additionally, as made precise in the next section, we consider a different limiting sequence, which accounts for the correlation

induced terms in the discrete variance.

In a first step, we reduce the dimensionality of the Monte-Carlo pricing scheme by eliminating the need to simulate the path of the log-returns. In a second step, a further simplification makes possible to avoid the simulation of the instantaneous variance path, by simulating directly from the distribution of the integrated continuous variance. Most importantly, a variation on the latter approach – termed hereafter the conditional Black-Scholes scheme – leads to an explicit discretization adjustment term, for which two simple analytic representations are provided in equations (23) and (24). This explicit discretization adjustment term is easily computable by standard Fourier transform methods, in any stochastic volatility model with a closed-form expression for the characteristic function of continuously sampled variance, e.g. Heston(1993) or the 3/2 model in Lewis(2000) and Carr, Sun(2007).

The rest of the paper is organized as follows. In Section 2, the core section, we develop the theoretical results and also define the proposed numerical schemes, specifically, the Monte Carlo and the Fourier based methods. Section 3 includes an overview of the numerical methods analyzed and Section 4 presents the performance of the different pricing schemes. The final section summarizes the conclusions. Proofs not given in the main text, as well as additional numerical results, can be found in the Appendix.

2 Options on discretely sampled variance

In this section we study the magnitude of the discretization effect in valuing options on realized variance.

2.1 Model

We start by setting a general stochastic volatility framework. We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ satisfying the usual conditions and let $(B_t, W_t)_{t\geq 0}$ be a two-dimensional Brownian motion with correlation ρ. Additionally, let $(N_t)_{t\geq 0}$ be an independent Poisson process with constant intensity λ. Assume that the stock price and its instantaneous variance $(S_t, v_t)_{t\geq 0}$ satisfy the following dynamics under a pricing measure $\mathbb{Q}$:

$$\frac{dS_t}{S_{t^-}} = (r - \delta - \lambda\mathbb{E}(e^{Z_t} - 1))dt + \sqrt{v_t}dB_t + (e^{Z_t} - 1)dN_t \tag{1}$$

$$dv_t = a(v_t)dt + b(v_t)dW_t \tag{2}$$

where $a : \mathbb{R}_+ \to \mathbb{R}$ and $b : \mathbb{R}_+ \to \mathbb{R}$ are Borel measurable functions such that the two-dimensional SDE (1), (2), admits a unique and non-exploding solution $(S_t, v_t)_{t\geq 0}$. The jump size in the log-return at time t is denoted by Z_t; jump sizes are assumed to be independent and identically distributed and to follow a normal distribution $\mathcal{N}(\mu_Z, \sigma_Z^2)$. The risk free interest rate, dividend

yield and correlation parameters are denoted by r, δ and ρ respectively. A large number of stochastic volatility models proposed in the literature belong to this framework. Important examples include Scott(1987), Heston(1993) and the 3/2 model discussed in Lewis(2000) and Carr, Sun(2007).

Consider a finite maturity $T > 0$ and let $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ be an equally spaced partition of $[0, T]$ with step-size $\Delta = \frac{T}{n}$. The (annualized) discretely sampled variance of $\log(S_t)$ over $[0, T]$ is defined as

$$RV_n = \frac{1}{T} \cdot \sum_{i=1}^{n} \log^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right). \tag{3}$$

A standard result in stochastic calculus (see, for example, Revuz, Yor (1999)) establishes that, as $n \to \infty$, RV_n converges in probability to the continuously sampled variance (or quadratic variation) of $\log(S_t)$ over the interval $[0, T]$ and scaled by the annualization factor $\frac{1}{T}$. Specifically, in our setup we have $\text{plim}_{n\to\infty} RV_n = \frac{1}{T}[\log(S_t)]_T = \frac{1}{T}\left(\int_0^T v_t dt + \sum_{0\le t\le T} Z_t^2 \Delta N_t\right)$, where plim denotes the limit in probability. In practice, variance contracts must be specified by using discrete sampling. For example, a call option on realized variance with maturity T and volatility strike σ_K delivers, at time T, the payoff:

$$VN \cdot \left(\frac{1}{T} \cdot \sum_{i=1}^{n} \log^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) - \sigma_K^2 \right)_+$$

where VN is a constant known as the variance notional; throughout this section, we set $VN = 1$. Our goal here is to compare the prices of options on discretely sampled variance to those of options on quadratic variation. An application of Itô's lemma to $\log(S_t)$ gives

$$d\log(S_t) = (r - \delta - \frac{v_t}{2} - \lambda\mathbb{E}[e^Z - 1])dt + \sqrt{v_t}dB_t + Z_t dN_t$$

from where, each discrete log-return can be written as

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) = (r-\delta-\lambda\mathbb{E}[e^Z-1])\cdot\frac{T}{n} - \frac{1}{2}\int_{t_{i-1}}^{t_i} v_s ds + \int_{t_{i-1}}^{t_i} \sqrt{v_s} dB_s + \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} Z_k$$

for all $i \in \{1, 2, \ldots, n\}$. In what follows we denote by $\mathcal{F}_t^W$ the filtration generated by the Brownian motion W_t driving the variance diffusion v_t; we recall that the process v_t serves to model the (stochastic) instantaneous variance of the asset price. A key observation is that, conditional on $\mathcal{F}_T^W$, the log-returns $\log\left(\frac{S_{t_i}}{S_{t_{i-1}}^-}\right)$, $i = \{1, 2, \ldots, n\}$, form a sequence of independent normally distributed (but not identically) random variables.

Using that

$$E\left[\sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} Z_k\right] = \lambda\frac{T}{n}\mu_Z$$

and

$$\begin{aligned} Var\left(\sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} Z_k\right) &= E\left[Var\left(\sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} Z_k \middle| N_{t_i} - N_{t_{i-1}}\right)\right] + \\ & \quad Var\left(E\left[\sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} Z_k \middle| N_{t_i} - N_{t_{i-1}}\right]\right) \\ &= \lambda\frac{T}{n}\sigma_Z^2 + \lambda\frac{T}{n}\mu_Z^2, \end{aligned}$$

we can conclude that:

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\Bigg|_{\mathcal{F}_T^W} \sim N\left(\mu_{n,i}, \sigma_{n,i}^2\right) \tag{4}$$

with means and variances $\mu_{n,i}$ and $\sigma_{n,i}^2$ given by

$$\begin{aligned} \mu_{n,i} &= \left(r - \delta - \lambda(e^{\mu_Z + \frac{1}{2}\sigma_Z^2} - \mu_Z - 1)\right) \cdot \frac{T}{n} - \frac{1}{2}\int_{t_{i-1}}^{t_i} v_s ds + \rho\int_{t_{i-1}}^{t_i} \sqrt{v_s} dW_s \\ \sigma_{n,i}^2 &= (1-\rho^2)\int_{t_{i-1}}^{t_i} v_s ds + (\mu_Z^2 + \sigma_Z^2)\lambda\frac{T}{n}. \end{aligned}$$

for all $n \geq 1$.

The result in (4) follows immediately from the property that the Brownian integral of any deterministic, locally-bounded function is a Gaussian process (see, for example, Revuz , Yor(1999)) and that jump sizes form an independent normally distributed sequence. We note that, in the absence of jumps, conditional on $\mathcal{F}_T^W$, the continuously sampled variance $\frac{1}{T}\int_0^T v_t dt$ is just a constant, whereas the discretely sampled variance RV_n still has a residual randomness driven by B_t. In practice, for typical parameter values, this residual randomness is not negligible and can lead to substantially higher prices for options on discretely sampled variance, especially for maturities less than one year.

2.2 Distribution of RV_n conditional on $\mathcal{F}_t^W$

We formulate here the main theorem which plays a key role in the numerical schemes proposed.

Theorem 2.1. *Conditional on $\mathcal{F}_T^W$, the discretely sampled realized variance*

$$\mathrm{RV}_n = \frac{1}{\Delta \cdot n} \cdot \sum_{i=1}^{n} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$$

converges in distribution to a normal random variable. More precisely, as $n \to \infty$, we have

$$\frac{n \cdot \Delta}{s_n}\left(\mathrm{RV}_n - \frac{\sum_{i=1}^{n} \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta}\right) \xrightarrow{d} N(0,1) \tag{5}$$

where

$$s_n^2 = \sum_{i=1}^{n} 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2.$$

Remark: Theorem 2.1 shows that the *conditional* distribution of RV_n is asymptotically normal $N\left(M_n, \Sigma_n^2\right)$, which will prove useful in calculating the price of options on realized variance. The conditional moments are given by

$$\begin{aligned} M_n &= \frac{\sum_{i=1}^{n} \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta} \\ \Sigma_n^2 &= \frac{s_n^2}{n^2 \cdot \Delta^2}. \end{aligned}$$

We emphasize that these moments are conditional on $\mathcal{F}_t^W$ – or alternatively – they hold given a known path of the instantaneous variance up to maturity.

Proof of Theorem 2.1 To establish this result, we use a generalized version of the central limit theorem (CLT) for triangular arrays of unequal components. To see why this is necessary, note that for each $n \geq 1$, in the expression of RV_n, we have a different sequence of squared log-returns and the components of each sequence have different variances. Specifically, we shall use the *Lindeberg-Feller* generalized CLT (see, for example, Ferguson(1996)) as formulated in Theorem 6.1 in Appendix 6.

Define the triangular sequence $Y_{n,i} = X_{n,i}^2 - \left(\mu_{n,i}^2 + \sigma_{n,i}^2\right)$, where $X_{n,i} = \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$ is the triangular sequence of log-returns. We seek to apply the Lindeberg-Feller Theorem 6.1 to the sequence $Y_{n,i}$. By construction, the condition $E(Y_{n,i}) = 0$ is satisfied.

Making use of the conditional normality of the log-returns $X_{n,i}$ we obtain, conditionally on $\mathcal{F}_T^W$:

$$\mathbb{E}(Y_{n,i}^2) = 2\sigma_{n,i}^4 + 4\mu_{n,i}^2\sigma_{n,i}^2.$$

The computation of this expectation follows from the straightforward (but tedious) use of the higher moments of the normal distribution.

Take any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. By the local properties of Brownian paths (see, for example, Revuz, Yor(1999)), the function $h(t) = \int_0^t \sqrt{v_s} dW_s$ is Hölder continuous with index α on $[0, T]$. We conclude that there exists a positive constant $K_1 > 0$, independent of n, such that

$$\left| \int_{t_{i-1}}^{t_i} \sqrt{v_s} dW_s \right| = \left| h\left(t_i\right) - h\left(t_{i-1}\right) \right| \leq K_1 \cdot \left| t_i - t_{i-1} \right|^{\alpha} = K_1 \cdot \frac{T^{\alpha}}{n^{\alpha}}$$

for all $n \geq 1$. Similarly, since the function $g(t) = \int_0^t v_s ds$ is Lipschitz continuous on $[0, T]$, there exists a positive constant $K_2 > 0$, independent of n, such that

$$\left| \int_{t_{i-1}}^{t_i} v_s ds \right| = \left| g\left(t_i\right) - g\left(t_{i-1}\right) \right| \leq K_2 \cdot \left| t_i - t_{i-1} \right| = K_2 \cdot \frac{T}{n}$$

for all $n \geq 1$. From the definition of $\mu_{n,i}$, we see that for all positive integers $n \geq T$, the following bound holds:

$$\begin{aligned} \left|\mu_{n,i}\right| &\leq \left| r - \delta - \lambda(e^{\mu_Z + \frac{1}{2}\sigma_Z^2} - \mu_Z - 1) \right| \cdot \frac{T}{n} + \frac{1}{2} K_2 \cdot \frac{T}{n} + |\rho| K_1 \cdot \frac{T^{\alpha}}{n^{\alpha}} \\ &\leq \left(\left| r - \delta - \lambda(e^{\mu_Z + \frac{1}{2}\sigma_Z^2} - \mu_Z - 1) \right| + \frac{1}{2} K_2 + |\rho| K_1 \right) \cdot \frac{T^{\alpha}}{n^{\alpha}} \\ &= C_1 \cdot \frac{T^{\alpha}}{n^{\alpha}}. \end{aligned} \tag{6}$$

Similarly, for $\sigma_{n,i}^2$ we obtain

$$\sigma_{n,i}^2 \leq (1 - \rho^2) \cdot K_2 \cdot \frac{T}{n} + (\mu_Z^2 + \sigma_Z^2)\lambda \frac{T}{n} = C_2 \cdot \frac{T}{n}. \tag{7}$$

We now show that the sequence $Y_{n,i}$ satisfies the sufficient condition of Lyapunov (28) for $\delta = 2$. Specifically, we show that

$$\lim_{n \to \infty} \frac{1}{s_n^4} \sum_{i=1}^{n} \mathbb{E}\left(Y_{n,i}^4\right) = 0$$

where

$$s_n^2 = \sum_{i=1}^{n} \mathbb{E}\left(Y_{n,i}^2\right) = \sum_{i=1}^{n} 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2.$$

Similar to the computation of the second conditional moment of $Y_{n,i}$, we can calculate its fourth moment as follows:

$$\mathbb{E}\left(Y_{n,i}^4\right) = 60\sigma_{n,i}^8 + 240\sigma_{n,i}^6 \cdot \mu_{n,i}^2 + 48\sigma_{n,i}^4 \cdot \mu_{n,i}^4 + 4\sigma_{n,i}^2 \cdot \mu_{n,i}^6. \tag{8}$$

Using inequalities (6), (7) we have for all positive integers $n \geq T$

$$\begin{aligned}\mathbb{E}\left(Y_{n,i}^4\right) &\leq 60 \cdot C_2^4 \cdot \frac{T^4}{n^4} + 240 \cdot C_2^3 \cdot C_1^2 \cdot \frac{T^{3+2\alpha}}{n^{3+2\alpha}} + 48 \cdot C_2^2 \cdot C_1^4 \cdot \frac{T^{2+4\alpha}}{n^{2+4\alpha}} + \\ &\quad 4 \cdot C_2 \cdot C_1^6 \cdot \frac{T^{1+6\alpha}}{n^{1+6\alpha}} \leq C_3 \cdot \frac{T^{1+6\alpha}}{n^{1+6\alpha}}.\end{aligned}$$

where we have used the fact that $1 + 6\alpha < 2 + 4\alpha < 3 + 2\alpha < 4$ and $\frac{T}{n} \leq 1$. By a simple application of the classic Cauchy-Schwarz (or, alternatively, Jensen's) inequality, we also obtain:

$$s_n^2 \geq 2 \sum_{i=1}^n \sigma_{n,i}^4 \geq \frac{2}{n} \left(\sum_{i=1}^n \sigma_{n,i}^2 \right)^2 = \frac{2}{n} \left(1 - \rho^2\right)^2 \left(\int_0^T v_t dt \right)^2 = \frac{C_4}{n}$$

where $C_4 > 0$ does not depend on n. Finally, this gives

$$\frac{1}{s_n^4} \sum_{i=1}^n \mathbb{E}\left(Y_{n,i}^4\right) \leq \frac{n \cdot C_3 \cdot \frac{T^{1+6\alpha}}{n^{1+6\alpha}}}{\frac{C_4^2}{n^2}} = \frac{C_3 \cdot T^{1+6\alpha}}{C_4^2} \cdot \frac{1}{n^{6\left(\alpha - \frac{1}{3}\right)}} \to 0 \text{ as } n \to \infty$$

where we have used that $\alpha > \frac{1}{3}$. By the Lindeberg-Feller Theorem 6.1, we conclude

$$\frac{n \cdot \Delta}{s_n} \left(\mathrm{RV}_n - \frac{\sum_{i=1}^n \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta} \right) = \frac{\sum_{i=1}^n Y_{n,i}}{s_n} \xrightarrow{d} N(0,1).$$

□

2.3 Pricing of options on discretely sampled variance

Theorem 2.1 and basic properties of normal random variables give that, conditional on $\mathcal{F}_T^W$, the (undiscounted) value of the discrete variance call can be approximated by

$$C^n\left(\sigma_K\right) = \Sigma_n \cdot \phi\left(\frac{M_n - \sigma_K^2}{\Sigma_n} \right) + \left(M_n - \sigma_k^2\right) \cdot N\left(\frac{M_n - \sigma_K^2}{\Sigma_n} \right) \tag{9}$$

where $\phi(\cdot)$ and $N(\cdot)$ denote the density and distribution functions of the standard normal law.

As a result of this formula, given an instantaneous variance path, one can calculate M_n and Σ_n and derive the conditional option price using equation (9). The results derived so far indicate the following method to price options on discrete variance by eliminating the need to simulate paths of the spot price S_t: for each simulated instantaneous variance path, compute the conditional option price by (9) and, finally, average over these conditional

prices. This will be the basis of two methods we will investigate for pricing options on realized variance, namely the *conditional normal* and *simplified conditional normal* methods. Later in the section, we explore ways to further simplify this conditional scheme by eliminating the need to compute the quantities $\mu_{n,i}$ and $\sigma^2_{n,i}$ for each variance path.

We remark that, on the purely theoretical front, it is possible to draw further on the tools of the generalized CLT to establish bounds on the approximation (9). These bounds are available in section 7 of the appendix.

2.4 Example of the Black-Scholes model

The standard and most basic model which fits in our framework is the Black-Scholes(1973) model, which will prove useful in our simplified conditional pricing schemes. In the standard Black-Scholes framework, we set $v_t = \sigma^2$ (a positive constant) and the log-returns now become i.i.d. normal:

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \sim N\left(\left(r-\delta-\frac{\sigma^2}{2}\right)\cdot\frac{T}{n}, \sigma^2\cdot\frac{T}{n}\right)$$

which gives that the distribution of RV_n satisfies

$$RV_n \stackrel{d}{=} \frac{\sigma^2}{n}\cdot\sum_{i=1}^{n}\left(\frac{r-\delta-\frac{\sigma^2}{2}}{\sigma}\cdot\sqrt{\frac{T}{n}}+Z_i\right)^2$$

where Z_i, with $1 \le i \le n$, here denotes a sequence of independent standard normal variables. We obtain that $RV_n \stackrel{d}{=} \frac{\sigma^2}{n}\cdot\chi'(n,\lambda)$ where $\chi'(n,\lambda)$ denotes the non-central chi-square distribution with n degrees of freedom and non-centrality parameter λ given by:

$$\lambda = \frac{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 T}{\sigma^2}. \tag{10}$$

A well-known result in mathematical statistics (see, for example, the classic treatment in Muirhead(2005)) establishes the following convergence in distribution to a standard normal:

$$\frac{\chi'(n,\lambda)-(n+\lambda)}{\sqrt{2(n+2\lambda)}} \stackrel{d}{\to} N(0,1)$$

as the number of degrees of freedom $n \to \infty$. Using the value of λ from (10), simple algebraic computations show that the distribution of RV_n converges to a normal distribution with mean and variance given by:

$$N\left(\sigma^2+\frac{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 T}{n}, \frac{2\sigma^4}{n}+\frac{4\left(r-\delta-\frac{\sigma^2}{2}\right)^2\sigma^2 T}{n^2}\right). \tag{11}$$

This is a special case of our more general result in Theorem 2.1. It is obtained using that, in the standard Black-Scholes model, $\mu_{n,i} = \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{T}{n}$ and $\sigma_{n,i}^2 = \sigma^2 \cdot \frac{T}{n}$ which leads to:

$$\begin{aligned} M_n &= \sigma^2 + \frac{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 T}{n} \\ \Sigma_n &= \sqrt{\frac{2\sigma^4}{n} + \frac{4\left(r - \delta - \frac{\sigma^2}{2}\right)^2 \sigma^2 T}{n^2}} \end{aligned}$$

as given in (11).

In the Black-Scholes model it is possible to derive an exact closed-form formula for the price of options on discrete variance. We formulate this result in Lemma 2.1 and note that it will be used in our conditional Black-Scholes scheme introduced later.

Lemma 2.1. *In the Black-Scholes model with constant volatility σ, we have*

$$\begin{aligned} \mathbb{E}\left(RV_n - \sigma_K^2\right)_+ &= \sigma^2 \cdot \left(1 - F_{\chi'}\left(\frac{\sigma_K^2 \cdot n}{\sigma^2}; \lambda, n+2\right)\right) \\ &\quad + \frac{\sigma^2 \cdot \lambda}{n} \cdot \left(1 - F_{\chi'}\left(\frac{\sigma_K^2 \cdot n}{\sigma^2}; \lambda, n+4\right)\right) \\ &\quad - \sigma_K^2 \cdot \left(1 - F_{\chi'}\left(\frac{\sigma_K^2 \cdot n}{\sigma^2}; \lambda, n\right)\right) \end{aligned} \tag{12}$$

where $F_{\chi'}(\cdot; \lambda, n)$ denotes the non-central chi-square CDF with n degrees of freedom and non-centrality parameter λ; the value of λ is given by (10).

Proof See Appendix. □.

2.5 Option pricing in a general stochastic volatility model

We now return to the general stochastic volatility case and seek to derive further simplified versions of the conditional pricing scheme implied by Theorem 2.2 and approximation (9). We note that, in a separate study, Sepp(2012) explores a different approach to adjusting for the discretization effect. Specifically, the discretization effect is treated independently by making the following approximation, in distribution:

$$RV_n \overset{d}{\simeq} \frac{1}{T}\int_0^T v_t dt - E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T v_t dt\right) + RV_n^{BS} \tag{13}$$

where RV_n^{BS} is the discretely sampled variance in an independent Black-Scholes model with time-dependent volatility $\sigma(t)^2 = E^{\mathbb{Q}}(v_t)$; we remark

that, in the presence of jumps, equation (13) involves a small modification and we refer the reader to Sepp(2012) for details. Given that, in a Black-Scholes model, the Fourier-Laplace transform of discretely sampled variance is easily obtained in closed-form and using the independence assumption, we can approximate the transform of RV_n:

$$L(\lambda) = E^{\mathbb{Q}}\left(e^{-\lambda \cdot RV_n}\right) \simeq e^{\lambda \cdot M} \cdot L_{QV}(\lambda) \cdot L_{RV_n^{BS}}(\lambda)$$

where $M = E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T v_t dt\right)$ and $L_{QV}(\lambda)$ is the transform of continuously sampled variance. As in several stochastic volatility models $L_{QV}(\lambda)$ is known in closed-form, this approach is attractive from the standpoint of using semi-analytical transform techniques and provides good accuracy for near the at-the-money region.

We will propose an alternative, transform-based approach, which does not rely on an independence assumption. The advantage will be that it leads to improved accuracy for out-of-the-money options. The magnitude of the discretization effect depends on the realization of continuously sampled variance; the smaller (larger) the latter, the smaller (larger) the discretization effect. Ignoring this dependence, will tend to overprice out-of-the-money variance puts and underprice out-of-the-money variance calls.

The following lemma will prove useful in our calculations. It was given in Barndorff-Nielsen, Shephard(2002), under the assumption of a variance process of finite variation. This would be unsuitable for our dynamics (2) of v_t but, fortunately, the assumption can be removed and, hence, we modify its statement accordingly.

Lemma 2.2. *Let $(v_t)_{t\geq 0}$ be a process which is a.s. locally bounded and has at most a countable number of discontinuity points on every finite interval. Then, for any fixed $T > 0$ and positive integer $k \in \mathbb{N}\backslash\{0\}$ we have*

$$\frac{n^{k-1}}{T^{k-1}} \cdot \sum_{i=1}^{n} \left(\int_{\frac{(i-1)T}{n}}^{\frac{iT}{n}} v_t dt\right)^k \to \int_0^T v_t^k dt \qquad a.s.$$

as $n \to \infty$.

Proof The proof is identical to Barndorff-Nielsen, Shephard (2002) except that we do not require the process $(v_t)_{t\geq 0}$ to be of locally bounded variation. The argument only requires that v_t^k be (a.s.) Riemann integrable on $[0, T]$, a condition which is satisfied under our assumptions for Lemma 2.2 (by, for example, Theorem 6.10 in Rudin (1976)).

□

Recall that, by Theorem 2.1, the conditional distribution of RV_n (i.e. given a path of the instantaneous variance) satisfies $\left(RV_n \middle| \mathcal{F}_T^W - M_n\right)/\Sigma_n \xrightarrow{d}$

$N(0,1)$ where

$$\begin{aligned} M_n &= \frac{\sum_{i=1}^n \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta} \\ \Sigma_n^2 &= \frac{s_n^2}{n^2 \cdot \Delta^2} = \frac{\sum_{i=1}^n 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2}{n^2 \cdot \Delta^2}. \end{aligned}$$

The simplified conditional schemes will set the variance-spot correlation ρ to zero. It turns out that setting $\rho = 0$ has little material impact on the prices of discrete variance options. To see this intuitively, note that, in the limit of continuous sampling, the correlation parameter plays no role in the price of variance options. In the numerical examples, we will see that for typical market parameters with strongly negative correlation, the simplified schemes perform very well.

We next want to apply Lemma 2.2 to obtain an approximation for M_n and Σ_n^2. Fix the following notations for real sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$: write $a_n = o(b_n)$ iff $\lim_{n\to\infty} |a_n/b_n| = 0$ and $a_n = O(b_n)$ iff $\limsup_{n\to\infty} |a_n/b_n| < \infty$. Firstly, observe that our instantaneous variance process $(v_t)_{t\in[0,T]}$, having a.s. continuous paths, clearly satisfies the assumptions of Lemma 2.2. Hence, letting $I_k = \frac{1}{T}\int_0^T v_t^k dt$ and recalling that, for $\rho = 0$, $\sigma_{n,i}^2 = \int_{t_{i-1}}^{t_i} v_t dt$, we obtain by Lemma 2.2:

$$\frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^4 = \frac{T}{n} \cdot I_2 + o\left(\frac{1}{n}\right) \tag{14}$$

$$\frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^6 = \frac{T^2}{n^2} \cdot I_3 + o\left(\frac{1}{n^2}\right). \tag{15}$$

Also, we note that $\frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^2 = I_1$. Expanding $\mu_{n,i}^2$, we have:

$$\begin{aligned} \mu_{n,i}^2 &= (r-\delta)^2 \cdot \frac{T^2}{n^2} - (r-\delta)\frac{T}{n} \cdot \int_{t_{i-1}}^{t_i} v_t dt + \frac{1}{4}\left(\int_{t_{i-1}}^{t_i} v_t dt\right)^2 \\ &= (r-\delta)^2 \cdot \frac{T^2}{n^2} - (r-\delta)\frac{T}{n} \cdot \sigma_{n,i}^2 + \frac{1}{4}\sigma_{n,i}^4 \end{aligned}$$

and, writing T for $n \cdot \Delta$, we obtain

$$\begin{aligned} M_n &= \frac{1}{T}\sum_{i=1}^n \mu_{n,i}^2 + \frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^2 \\ &= (r-\delta)^2\frac{T}{n} - (r-\delta)\frac{T}{n}I_1 + \frac{1}{4}\frac{T}{n}I_2 + o\left(\frac{1}{n}\right) + I_1 \\ &= I_1 + O\left(\frac{1}{n}\right) \end{aligned} \tag{16}$$

We next apply a similar approach to Σ_n^2. Using again the relations (14) , (15) obtained by Lemma 2.2, simple algebra gives:

$$\begin{aligned}
\Sigma_n^2 &= \frac{2}{T^2}\sum_{i=1}^{n}\sigma_{n,i}^4 + \frac{4}{T^2}\sum_{i=1}^{n}\mu_{n,i}^2\cdot\sigma_{n,i}^2 \\
&= \frac{2}{n}\cdot I_2 + o\left(\frac{1}{n}\right) + 4\left((r-\delta)^2\cdot\frac{T}{n^2}I_1 - (r-\delta)\frac{T}{n^2}I_2 + \frac{1}{4}\cdot\frac{T}{n^2}I_3 + o\left(\frac{1}{n^2}\right)\right) \\
&= \frac{2}{n}\cdot I_2 + o\left(\frac{1}{n}\right) \qquad (17)
\end{aligned}$$

From (16) and (17), we have obtained the following result for the conditional distribution of RV_n:

$$RV_n\Big|\mathcal{F}_T^W \sim N\left(I_1 + O\left(\frac{1}{n}\right), \frac{2}{n}\cdot I_2 + o\left(\frac{1}{n}\right)\right) \qquad (18)$$

The Conditional Normal Scheme. Keeping the leading order terms in (18), we obtain the approximation, in distribution:

$$RV_n\Big|\mathcal{F}_T^W \sim N\left(\frac{1}{T}\int_0^T v_t dt, \frac{2}{n}\cdot\frac{1}{T}\int_0^T v_t^2 dt\right). \qquad (19)$$

Remark: We note that by keeping the leading order terms, the expectation of the discrete realized variance in equation (19) is approximated with the integrated variance. However, it is known (see, e.g., Broadie, Jain(2008)) that the fair strike of a variance swap on the discrete realized variance becomes larger when the sampling period increases, which implies that the discrete realized variance has a larger expected value than the integrated variance. We analyze this effect in the numerical section.

By virtue of relation (19), we formulate the *conditional normal* pricing scheme as follows: (a) simulate a variance path v_t, $t \in [0,T]$ and compute I_1 and I_2, (b) price the conditional variance call by setting $M_n = I_1$ and $\Sigma_n = \sqrt{\frac{2}{n}\cdot I_2}$ in formula (9) and (c) average conditional prices by repeating (a), (b). Note that this approach, while no longer requiring to compute the quantities $\mu_{n,i}$ and $\sigma_{n,i}^2$, still requires the simulation of the entire variance path v_t on $[0,T]$ in order to allow us to extract both I_1 and I_2.

We notice that by Jensen's inequality $I_2 \geq I_1^2$ and hence the alternative approximation

$$RV_n\Big|\mathcal{F}_T^W \sim N\left(\frac{1}{T}\int_0^T v_t dt, \frac{2}{n}\left(\frac{1}{T}\int_0^T v_t dt\right)^2\right) \qquad (20)$$

which underestimates the variance term may cause some underpricing of options on realized variance (at least, relative to (19)). On the other hand,

this approximation will make it possible to simulate just from the law of integrated continuous variance $I_1 = \frac{1}{T}\int_0^T v_t dt$, without the need to generate the entire variance path v_t on $[0,T]$. We call this scheme the *Simplified Conditional Normal* scheme. We shall observe, in the numerical examples, that the underpricing is usually small.

The Simplified Conditional Black-Scholes Scheme. It can be shown that approximation (20) is asymptotically equivalent to assuming that the *conditional* pricing model is Black-Scholes. Specifically, conditional on a realization of the integrated continuous variance $\frac{1}{T}\int_0^T v_t dt$, suppose the model for the underlying price is Black-Scholes with variance parameter:

$$\sigma^2 = \frac{1}{T}\int_0^T v_t dt. \tag{21}$$

We have seen that, in a Black-Scholes model, the discretely sampled variance is non-central chi-square distributed $\chi^{'}(n,\lambda)$. In turn, keeping only the leading order terms in (11), we have that $\chi^{'}(n,\lambda)$ is approximately $N\left(\sigma^2, \frac{2}{n}\sigma^4\right)$. Replacing the value of σ^2 from (21), we see that the conditional Black-Scholes approach leads, in fact, to approximation (20).

By virtue of relation (20), we formulate the *simplified conditional Black-Scholes* (SCBS) scheme as follows: (a) simulate from the law of integrated continuous variance $I_1 = \frac{1}{T}\int_0^T v_t dt$, (b) price the conditional variance call by setting $\sigma^2 = I_1$ in the *exact* Black-Scholes formula of Lemma 2.1, (c) average conditional prices by repeating (a), (b). We note that the SCBS scheme does not require to simulate the entire path of v_t, $t \in [0,T]$.

Pursuing further the SCBS scheme, it is possible to derive a simple discretization adjustment requiring *no* Monte-Carlo simulation. Specifically, under the assumption (20) and provided the continuously sampled variance I_1 posses a Fourier transform in closed-form, we next derive a leading-order discretization adjustment based on a simple Fourier inversion. In the following, we regard the (undiscounted) prices of options on realized variance as functions of the variance strike $\mathcal{V} = \sigma_K^2$ and define $C_n, C : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} C_n(\mathcal{V}) &= 1_{\mathcal{V}\geq 0}\cdot \mathbb{E}\Big(RV_n - \mathcal{V}\Big)_+ \\ C(\mathcal{V}) &= 1_{\mathcal{V}\geq 0}\cdot \mathbb{E}\Big(I_1 - \mathcal{V}\Big)_+ \end{aligned}$$

where $I_1 = \frac{1}{T}\int_0^T v_t dt$ and, under the SCBS scheme (20), $RV_n \big| \mathcal{F}_T^W \sim N\left(I_1, \frac{2\cdot I_1^2}{n}\right)$. Assuming $\mathbb{E}(RV_n^2) < \infty$ and $\mathbb{E}(I_1^2) < \infty$, we first check that both functions $C_n(\cdot)$ and $C(\cdot) \in L^1(\mathbb{R})$, i.e. are integrable on $\mathbb{R}$. For example, we have for

$C_n(\mathcal{V})$:

$$\begin{aligned}\int_{-\infty}^{\infty}\left|C_n\left(\mathcal{V}\right)\right|d\mathcal{V} &= \int_0^{\infty}\mathbb{E}\left(RV_n-\mathcal{V}\right)_+ d\mathcal{V}\\ &= \mathbb{E}\int_0^{RV_n}\left(RV_n-\mathcal{V}\right)d\mathcal{V}=\frac{1}{2}\cdot\mathbb{E}\left(RV_n^2\right)<\infty\end{aligned}$$

where we interchanged integration and expectation as the integrand is non-negative; an identical argument holds for $C(\mathcal{V})$. Therefore, both functions will have well defined Fourier transforms, hereafter denoted by $\widehat{C}_n(u)$ and $\widehat{C}(u)$, respectively. The following formula, which first appeared in Carr et al. (2005), can be established for the Fourier transforms:

$$\begin{aligned}\widehat{C}_n(u) &= \int_{-\infty}^{\infty}e^{iu\mathcal{V}}\cdot C_n(\mathcal{V})d\mathcal{V}=\frac{1-\varphi_n(u)}{u^2}-i\cdot\frac{\mathbb{E}(RV_n)}{u}\\ \widehat{C}(u) &= \int_{-\infty}^{\infty}e^{iu\mathcal{V}}\cdot C(\mathcal{V})d\mathcal{V}=\frac{1-\varphi(u)}{u^2}-i\cdot\frac{\mathbb{E}(I_1)}{u}\end{aligned}$$

where $\varphi_n(u)=\mathbb{E}\left(e^{iuRV_n}\right)$ and $\varphi(u)=\mathbb{E}\left(e^{iuI_1}\right)$ denote the Fourier transforms of RV_n and I_1, respectively.

The key idea is to now consider the difference between the price of options on discrete variance and the price of options on continuous variance by defining the new function $\Lambda\left(\mathcal{V}\right)=C_n(\mathcal{V})-C(\mathcal{V})\in L^1\left(\mathbb{R}\right)$. Using that $\mathbb{E}\left(RV_n\right)=\mathbb{E}\left(\mathbb{E}\left(RV_n\middle|\mathcal{F}_T^W\right)\right)=\mathbb{E}\left(I_1\right)$ and by the linearity of the Fourier transform, we obtain

$$\widehat{\Lambda}(u)=\int_{-\infty}^{\infty}e^{iu\mathcal{V}}\cdot\Lambda(\mathcal{V})d\mathcal{V}=\widehat{C}_n(u)-\widehat{C}(u)=\frac{\varphi(u)-\varphi_n(u)}{u^2}.$$

The resulting discretization adjustment is given by

$$\widehat{\Lambda}(u)=\mathbb{E}\left(e^{iuI_1}\cdot\frac{1-e^{-\frac{u^2I_1^2}{n}}}{u^2}\right).$$

Expanding the second term of the product under the expectation and keeping the term of order $O\left(\frac{1}{n}\right)$, we can write

$$\widehat{\Lambda}(u)=\mathbb{E}\left(e^{iuI_1}\cdot\frac{I_1^2}{n}\right)+O\left(\frac{1}{n^2}\right):=\widehat{\Lambda}_1(u)+O\left(\frac{1}{n^2}\right). \tag{22}$$

Making use again of the assumption $\mathbb{E}(I_1^2)<\infty$, we have that $\varphi(u)$, the characteristic function of I_1, is twice continuously differentiable with respect to u and $\frac{\partial^2\varphi(u)}{\partial u^2}=-\mathbb{E}\left(e^{iuI_1}\cdot I_1^2\right)$. This gives that the leading term $\widehat{\Lambda}_1(u)$ in (22) can be written as

$$\widehat{\Lambda}_1(u)=-\frac{1}{n}\cdot\frac{\partial^2\varphi(u)}{\partial u^2}$$

We now proceed to determine the discretization adjustment term that results from considering only the term $\widehat{\Lambda}_1(u)$ in the expansion (22). More precisely, we compute

$$\begin{aligned}
\Lambda_1(\mathcal{V}) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\widehat{\Lambda}_1(u)du = \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\frac{1}{n}\cdot\frac{\partial^2\varphi(u)}{\partial u^2}du \\
&= \frac{-1}{2\pi n}\cdot\left[e^{-iu\mathcal{V}}\cdot\frac{\partial\varphi(u)}{\partial u}\bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\mathcal{V}\cdot e^{-iu\mathcal{V}}\cdot\frac{\partial\varphi(u)}{\partial u}du\right] \\
&= \frac{-1}{2\pi n}\cdot\left[e^{-iu\mathcal{V}}\cdot\frac{\partial\varphi(u)}{\partial u}\bigg|_{-\infty}^{\infty} + i\mathcal{V}\cdot e^{-iu\mathcal{V}}\cdot\varphi(u)\bigg|_{-\infty}^{\infty}\right. \\
&\qquad\qquad \left. -\int_{-\infty}^{\infty}\mathcal{V}^2\cdot e^{-iu\mathcal{V}}\cdot\varphi(u)du\right] \\
&= \frac{\mathcal{V}^2}{n}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\varphi(u)du
\end{aligned}$$

where both boundary terms, resulting from the integration by parts, will vanish by a simple application of the classical Riemann-Lebesgue lemma (see, for example, Feller (1991)). Finally, we obtain

$$\Lambda_1(\mathcal{V}) = \frac{\mathcal{V}^2}{n}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\varphi(u)du. \tag{23}$$

We notice that the computation of the discretization adjustment term $\Lambda_1(\mathcal{V})$ involves a simple Fourier inversion of $\varphi(u)$. Alternatively, the discretization adjustment can be written more compactly as

$$\Lambda_1(\mathcal{V}) = \frac{\mathcal{V}^2}{n}\cdot q\left(\mathcal{V}\right) \geq 0 \tag{24}$$

where $q\left(\mathcal{V}\right)$ denotes the density of the continuously sampled variance $I_1 = \frac{1}{T}\int_0^T v_t dt$. Both representations (23) and (24) provide a remarkably simple formula for the leading order discretization adjustment term.

For completeness, we note that (under the SCBS scheme) the Fourier transforms of RV_n and I_1 can also be linked as follows:

$$\begin{aligned}
\varphi_n(u) &= \mathbb{E}\left(e^{iuRV_n}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{iuRV_n}\middle|\mathcal{F}_T^W\right)\right) = \mathbb{E}\left(e^{iuI_1-\frac{u^2I_1^2}{n}}\right) \\
&= \mathbb{E}\left[e^{iuI_1}\left(1-\frac{u^2I_1^2}{n}+O\left(\frac{1}{n^2}\right)\right)\right] \\
&= \varphi(u) - u^2\mathbb{E}\left[e^{iuI_1}\frac{I_1^2}{n}\right] + O\left(\frac{1}{n^2}\right) \\
&= \varphi(u) + \frac{u^2}{n}\frac{\partial^2\varphi(u)}{\partial u^2} + O\left(\frac{1}{n^2}\right).
\end{aligned} \tag{25}$$

Formula (25) can be used to approximate the price of options on discrete realized variance by using it directly in a standard Fourier option pricer as in Sepp(2008, 2012).

We note that, as expected, the discretization adjustment in (23)-(24) is non-negative reflecting that options on discrete variance are more expensive than options on continuous variance. Therefore, if we work in a stochastic volatility model which admits a closed-form solution for $\varphi(u)$ (e.g. Heston(1993) or the 3/2 model in Carr, Sun(2007)), we first price options on continuously sampled variance using standard Fourier methods from the literature and then add the positive adjustment term (23)-(24), which is also computable by simple Fourier inversion.

3 Summary of methods

We summarize here the different numerical methods which will be used to compute the price of options on discretely sampled variance. The numerical performance of these methods will be detailed in Section 4.

- Continuous Sampling: Options on quadratic variation are priced using Fourier transform methods. The Fourier-Laplace Transform of the option price is inferred from the transform of the integrated variance and we use a Gamma distributed random variable as control variate as in Drimus(2012); the numerical inversion is performed using the technique of Iseger(2006). As the discretization effect is neglected, the prices of options on discrete realized variance are expected to be higher than those obtained with this method. This phenomenon was previously discussed in Bühler(2009), Gatheral(2008), Keller-Ressel, Muhle-Karbe(2010), Sepp(2012), and Keller-Ressel, Griessler(2012).

- Monte Carlo sampling: The prices of options on variance are obtained by Monte-Carlo simulation of the SDE (1), (2) by the technique in Andersen(2008): the simulation technique combines the exact simulation scheme of Broadie, Kaya(2006) with a numerically efficient local moment-matching method. We use the prices obtained with this method as reference prices; the number of Monte Carlo paths used is 150,000 and the step size is one business day ($dt = \frac{1}{252}$).

- Sepp(2012) approximation: Options are priced using an approximation in distribution of the discretely sampled variance as a function of the quadratic variation in a jump-diffusion stochastic volatility model and the discrete variance in an independent Black-Scholes model. The Fourier-Laplace transform of the discrete realized variance is obtained using equation (11) in Sepp(2012).

- Conditional Normal method: Options on discrete realized variance are priced by simulating paths of the variance process and computing the average of the *conditional* option prices over all paths. This method is based on the result that the law of realized variance, conditional on the variance path, is asymptotically normal by Theorem 2.1. The conditional moments of realized variance, M_n and Σ_n^2, are approximated by applying Lemma 2.2; conditional option prices are finally obtained using formula (9). The main drawback of this method is that it requires to simulate the entire path of the variance process.

- Simplified Conditional Normal method: Prices of options are obtained by simulating from the law of the integrated variance. The Laplace transform of the integrated variance in a CIR model is known in closed-form, see for example Cox et al.(1985) or Dufresne (2001). Fourier inversion allows recovering the cumulative distribution function of the integrated variance which, applied to uniformly distributed random variables, generates a sample from the law of integrated variance. The conditional option prices are calculated using formula (9). Compared to the conditional normal approach, this method does not require to simulate the entire path of the variance process.

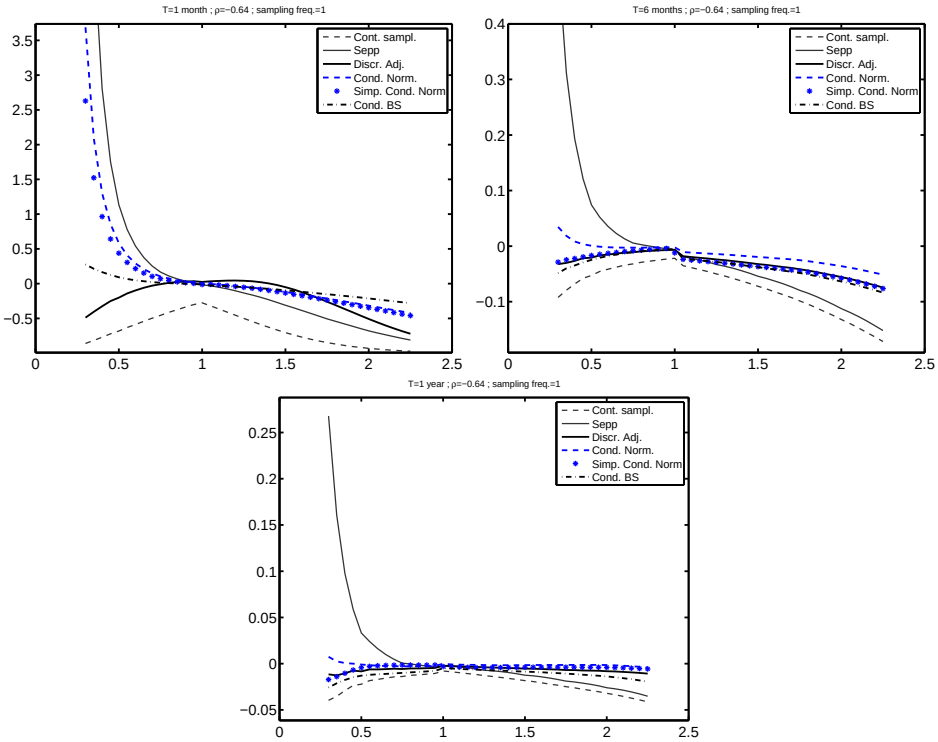
- Simplified Conditional Black-Scholes (SCBS): Prices of options on discrete realized variance are computed by simulating from the law of integrated variance and calculating the conditional option prices using the exact formula in Lemma 2.1. This approach is equivalent to assuming that the conditional model is Black-Scholes with variance parameter equal to the realization of the integrated continuous variance. As in the simplified conditional normal method, this approach does not require to simulate the entire path of the variance process.

- Fourier-based discretization adjustment: Prices of options on realized variance are represented as the sum of the price of an option on quadratic variation and a discretization adjustment term, both computed using Fourier techniques. The discretization adjustment term is computed using equation (23).

4 Numerical Examples

In this section, we consider a standard Heston(1993) model. We examine two different parameter sets. The reference set $\mathcal{P}_1$ results from the estimation in the study of Bakshi, Cao, Chen(1997) for the S&P500 index. The estimated parameters are $(v_0, k, \theta, \epsilon, \rho)_{\mathcal{P}_1} = (18.65\%^2, 1.15, 18.65\%^2, 0.39, -0.64)$. The second set $\mathcal{P}_2$ is a slight modification of the first one and aims to analyze the effect of a strongly negative correlation coefficient: $(v_0, k, \theta, \epsilon, \rho)_{\mathcal{P}_2} =$

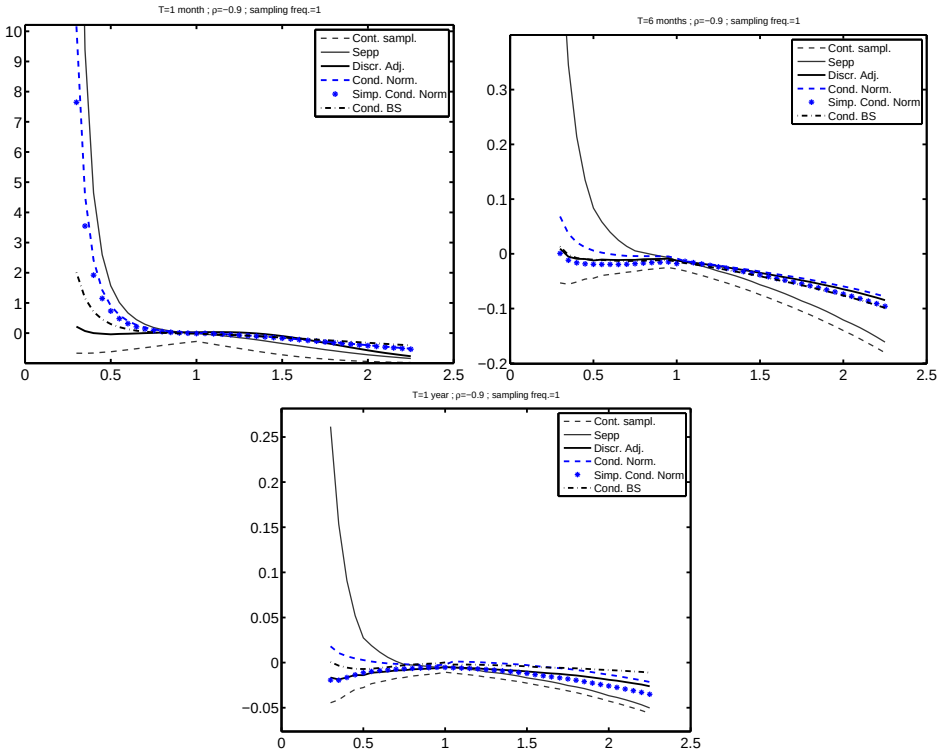
Figure 1: Relative errors in prices of OTM options on discrete realized variance using the parameter set $\mathcal{P}_1$. The sampling frequency is one day. The left part of the graphs (moneyness smaller than 1) corresponds to OTM puts whereas the right part (moneyness larger than 1) corresponds to OTM calls.



$(18.65\%^2, 1.15, 18.65\%^2, 0.39, -0.9)$. Indeed, Bernard, Cui(2014) show that the leverage coefficient plays an important role in the convergence of the fair strike of the discrete variance swap rate.

We analyze the performance of the different methods summarized in Section 3 in terms of out-of-the-money (OTM) option price error across a wide range of volatility strikes, for three different maturities (1 month, 6 months and one year) and for different sampling frequencies ranging from one to five days. The relative error of the methods as a function of the moneyness $\frac{\sigma_K^2}{v_0}$ is displayed in Figures 1, 2, 4, 3 which correspond to parameter sets $\mathcal{P}_1$ and $\mathcal{P}_2$ and different sampling frequencies. Note that the part of the figures which is to the left of moneyness 1 corresponds to the error made in pricing OTM put options, whereas the part to the right of moneyness 1 represents the error made in pricing OTM call options. This justifies the kink at moneyness equal to 1 in some of the figures. The average relative differences of the prices of each method compared to those of the Monte Carlo method

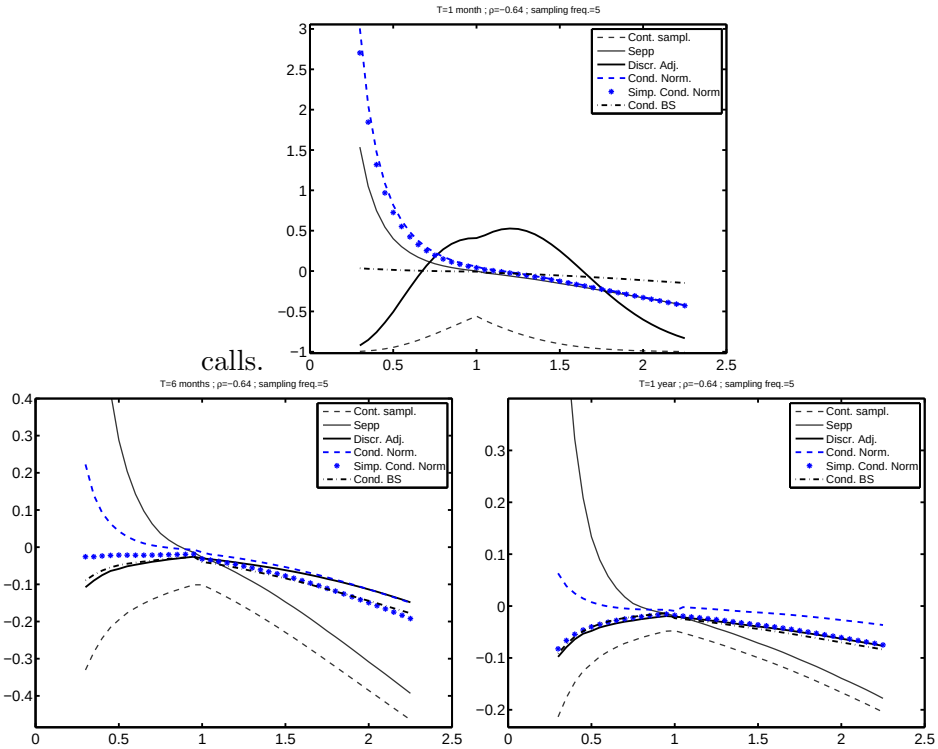
Figure 2: Relative errors in prices of OTM options on discrete realized variance using the parameter set $\mathcal{P}_2$ (more negative value of ρ). The sampling frequency is one day. The left part of the graphs (moneyness smaller than 1) corresponds to OTM puts whereas the right part (moneyness larger than 1) corresponds to OTM calls.



for different levels of moneynesses are listed in Tables 1 and 2. The upper part of the tables describes the results for a daily sampling frequency of the realized variance, whereas the lower part uses a sampling period of five days. Results for intermediate sampling frequencies are not displayed but available upon request. Tables 1 and 2 also report the computational time needed for each method to calculate the price for all strikes in seconds, on an Intel Core i7-2820QM CPU 2.30GHz.

As expected, we observe that the continuous sampling method underprices options for all strikes and parameter sets. Whereas the bias is rather small for longer maturities, it becomes non negligible for shorter maturities. This method is computationally much less intensive than Monte Carlo simulations, however, its performance is almost systematically outperformed by the Simplified Conditional Normal method (except for out-of-the-money puts with short times-to-maturities), whose computational time is also less than a second per option.

Relative errors in prices of OTM options on discrete realized variance using the parameter set $\mathcal{P}_1$. The sampling frequency is five days. The left part of the graphs (moneyness smaller than 1) corresponds to OTM puts whereas the right part (moneyness larger than 1) corresponds to OTM calls.



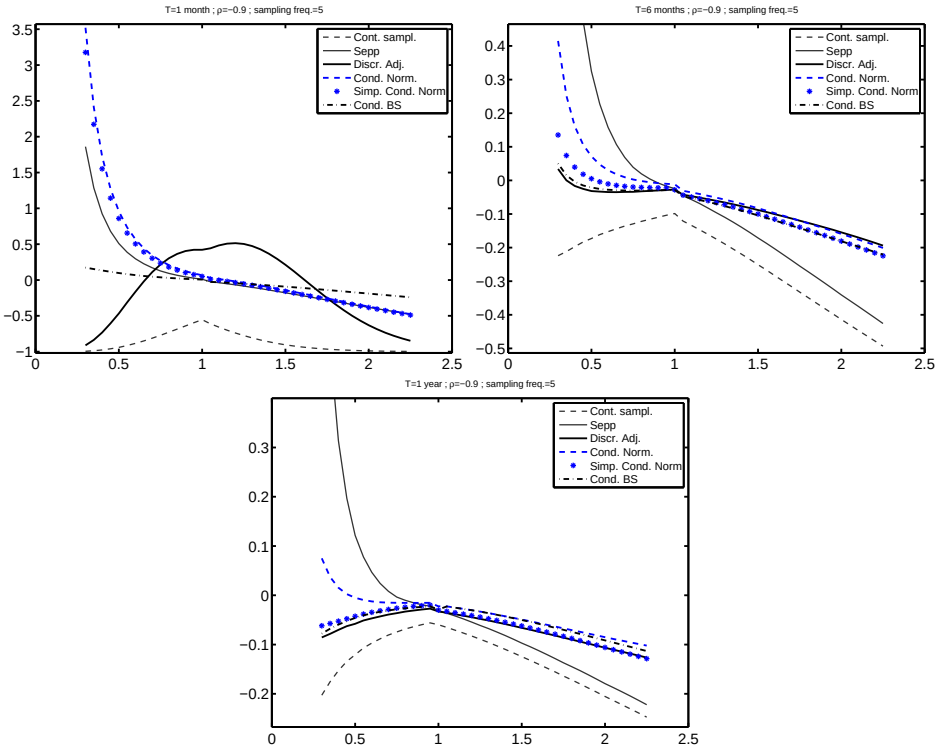
Sepp(2012)'s approximation performs well in the at-the-money (ATM) region but overprices deep out-of-the-money (OTM) put options and underprices deep out-the-money (OTM) call options. It is appealing near the ATM level and appears to be among the fastest methods considered.

For short maturities, we find that the Conditional Black-Scholes method is the one which performs the best. The Fourier-based discretization adjustment underprices OTM put options and the Conditional Normal methods overprices them. Both underprice OTM calls. The errors are magnified when the leverage coefficient is larger in absolute value, however the Conditional Black-Scholes method remains the preferred choice for short maturities, with very small relative errors.

When the maturity increases, we find that the best method becomes the Fourier-based discretization adjustment which is both fast and accurate.

We remark that in the right part of the smile all methods tend to slightly underprice options. As explained in Section 2, one of the reasons is that the simplified conditional methods approximate the expected value of discrete realized variance by the expected value of the integrated variance.

Figure 3: Relative errors in prices of OTM options on the discrete realized variance using the parameter set $\mathcal{P}_2$ (more negative value of ρ). The sampling frequency is five days. The left part of the graphs (moneyness smaller than 1) corresponds to OTM puts whereas the right part (moneyness larger than 1) corresponds to OTM calls.



Comparing the relative errors for two distinct values of ρ did not highlight a significant deterioration of the performance of the approximations when the leverage becomes more negative. Additionally, the same numerical experiments were performed also with a different suite of parameters: in particular we considered the case when $\theta > v_0$, which is often encountered in practice. The parameter sets become $(v_0, k, \theta, \epsilon, \rho)_{\mathcal{P}_3} = (15\%^2, 1.15, 20\%^2, 0.39, -0.64)$ and, combined with a strongly negative correlation coefficient: $(v_0, k, \theta, \epsilon, \rho)_{\mathcal{P}_4} = (15\%^2, 1.15, 20\%^2, 0.39, -0.9)$. The results remain qualitatively unchanged; results are reported in Appendix C.

In summary, for short-maturity options, we recommend using the Conditional Black-Scholes method. For mid-term to long-maturity options, we recommend using the Fourier-based discretization adjustment, which is fast and accurate.

Table 1: OTM option prices under parameter sets $\mathcal{P}_3$ and $\mathcal{P}_4$. "OTM put" refers to an option with moneyness (Strike/v_0) 0.6, "ATM" to an option with moneyness 1 and "OTM call" to an option with moneyness 1.5. Prices are in percentages. The computational time is in seconds.

	T = 1 month				T = 6 months				T = 1 year			
	OTM put	ATM	OTM call	Comp. time	OTM put	ATM	OTM call	Comp. time	OTM put	ATM	OTM call	Comp. time
1 - Daily sampling frequency												
Parameter set $\mathcal{P}_1$												
Monte Carlo	0.0902	0.6410	0.1971	42.2845	0.2763	0.9531	0.4534	243.1584	0.3371	1.0683	0.5837	542.8361
Cont. sampling	0.0369	0.4640	0.0583	0.0110	0.2666	0.9318	0.4268	0.0173	0.3299	1.0609	0.5745	0.0167
Sepp's approx.	0.1376	0.6312	0.1360	0.0250	0.2880	0.9465	0.4351	0.1703	0.3411	1.0669	0.5779	0.2316
Cond. Normal	0.1156	0.6429	0.1750	66.7809	0.2774	0.9502	0.4478	263.5809	0.3366	1.0707	0.5939	544.6329
Simp. Cond. Normal	0.1088	0.6343	0.1690	0.8447	0.2720	0.9437	0.4457	0.9117	0.3349	1.0618	0.5797	0.8300
Simp. Cond. BS	0.0922	0.6280	0.1777	130.1879	0.2742	0.9478	0.4471	144.9852	0.3333	1.0658	0.5820	161.9462
Discr. Adj.	0.0806	0.6574	0.1876	0.0122	0.2740	0.9464	0.4451	0.0108	0.3336	1.0669	0.5821	0.0345
Parameter set $\mathcal{P}_2$												
Monte Carlo	0.0824	0.6412	0.2081	16.5117	0.2785	0.9606	0.4601	97.9833	0.3371	1.0723	0.5879	214.1854
Cont. sampling	0.0369	0.4640	0.0583	0.0101	0.2666	0.9318	0.4268	0.1106	0.3299	1.0609	0.5745	0.7979
Sepp's approx.	0.1376	0.6312	0.1360	0.0171	0.2880	0.9465	0.4351	0.1690	0.3411	1.0669	0.5779	0.6648
Cond. Normal	0.1153	0.6425	0.1745	33.7436	0.2761	0.9488	0.4464	110.7486	0.3371	1.0699	0.5862	220.8680
Simp. Cond. Normal	0.1085	0.6329	0.1728	0.4430	0.2750	0.9412	0.4423	0.8939	0.3342	1.0666	0.5808	0.9599
Simp. Cond. BS	0.0929	0.6306	0.1776	44.3430	0.2766	0.9481	0.4390	48.6013	0.3352	1.0725	0.5854	56.6216
Discr. Adj.	0.0806	0.6574	0.1876	0.0074	0.2740	0.9464	0.4451	0.0159	0.3336	1.0669	0.5821	0.1029
2 - Sampling frequency every five days												
Parameter set $\mathcal{P}_1$												
Monte Carlo	0.3361	1.0499	0.5892	19.1224	0.3187	1.0367	0.5537	97.7155	0.3617	1.1140	0.6380	216.1839
Cont. sampling	0.0369	0.4640	0.0583	0.0116	0.2666	0.9318	0.4268	0.0123	0.3299	1.0609	0.5745	0.0096
Sepp's approx.	0.4129	1.0512	0.4994	0.0112	0.3642	1.0085	0.4720	0.0271	0.3824	1.0926	0.5925	0.0464
Cond. Normal	0.4988	1.1085	0.5197	21.2914	0.3244	1.0224	0.5240	46.4522	0.3614	1.1034	0.6301	77.9542
Simp. Cond. Normal	0.4787	1.0935	0.5148	0.5099	0.3118	1.0040	0.5108	0.5391	0.3504	1.0940	0.6150	0.6176
Simp. Cond. BS	0.3387	1.0431	0.5572	34.9388	0.3055	0.9958	0.5072	51.3458	0.3509	1.0879	0.6099	51.5603
Discr. Adj.	0.2666	1.4792	0.7368	0.0115	0.3036	1.0054	0.5194	0.0090	0.3483	1.0913	0.6128	0.0101
Parameter set $\mathcal{P}_2$												
Monte Carlo	0.3185	1.0401	0.6079	19.0544	0.3146	1.0338	0.5696	97.8267	0.3654	1.1284	0.6562	214.2530
Cont. sampling	0.0369	0.4640	0.0583	0.0098	0.2666	0.9318	0.4268	0.0078	0.3299	1.0609	0.5745	0.0132
Sepp's approx.	0.4129	1.0512	0.4994	0.0104	0.3642	1.0085	0.4720	0.0283	0.3824	1.0926	0.5925	0.0465
Cond. Normal	0.4991	1.1091	0.5213	24.8379	0.3241	1.0217	0.5224	46.8337	0.3609	1.1031	0.6238	79.5657
Simp. Cond. Normal	0.4792	1.0918	0.5119	0.5262	0.3113	1.0049	0.5126	0.5654	0.3528	1.0945	0.6153	0.6284
Simp. Cond. BS	0.3402	1.0468	0.5519	32.2521	0.3058	1.0046	0.5103	50.0830	0.3520	1.0960	0.6233	48.9897
Discr. Adj.	0.2666	1.4792	0.7368	0.0096	0.3036	1.0054	0.5194	0.0095	0.3483	1.0913	0.6128	0.0096

5 Conclusion

Discrete sampling has significant impact on the valuation of options on realized variance. We start by providing a characterization of this effect in Theorem 2.1 under general stochastic volatility dynamics and construct several methods for pricing options on discrete variance.

We remark among the various methods, the conditional Black-Scholes method and the Fourier-based discretization adjustment; the latter leads to the remarkably simple and tractable leading-order discretization adjustment term obtained in equation (23). The result can be implemented in any stochastic volatility model which admits a closed-form expression for the Fourier transform of continuously sampled variance; important examples of models where our result is directly applicable include both affine stochastic volatility models (e.g. Heston (1993)) as well as tractable non-affine models, such as the 3/2 model in Lewis (2000) and Carr, Sun (2007).

Finally, we perform an extensive numerical study which compares the performance of the different approximations. Based on accuracy and computational time, we show that the conditional Black-Scholes scheme performs the best for short-maturity options, whereas we recommend using the Fourier-based discretization adjustment for mid- to long-maturity options.

6 Appendix A

As a lemma of Theorem 2.1 we shall use the *Lindeberg-Feller* generalized CLT (see, for example, Ferguson(1996)) as formulated in Theorem 6.1.

Theorem 6.1 (Generalized CLT: Lindeberg-Feller)**.** *Let $Z_{n,i}$, $n = 1, 2, \ldots$ $i = 1, 2, \ldots, n$ be a triangular sequence of random variables such that $E(Z_{n,i}) = 0$, $E\left(Z_{n,i}^2\right) < \infty$ and for each fixed $n = 1, 2, \ldots$ the random variables $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$ are independent. If the* Lindeberg condition *is satisfied i.e. for all $\epsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}\left(|Z_{n,i}|^2; |Z_{n,i}| > \epsilon \cdot s_n\right) = 0 \tag{26}$$

where $s_n^2 = \sum_{i=1}^{n} \mathbb{E}\left(Z_{n,i}^2\right)$, then we have the convergence in distribution

$$\frac{Z_{n,1} + Z_{n,2} + \ldots + Z_{n,n}}{s_n} \xrightarrow{d} N(0,1). \tag{27}$$

To verify the Lindeberg condition (26), it is usually easier to check the sufficient condition of *Lyapunov* (see, for example, Petrov (1995)). Specifi-

cally, if there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} E\left(|Z_{n,i}|^{2+\delta}\right) = 0 \tag{28}$$

then the conclusion of the Lindeberg-Feller theorem (27) holds.

7 Appendix B

To establish bounds on the approximation (9), we use the following generalization of the *Berry-Essen* inequalities due to Bikelis(1966), and which can also be found in Petrov(2007).

Theorem 7.1 (Bikelis)**.** *Assume $Z_1, Z_2, \ldots, Z_n$ are independent random variables with mean zero and $E\left(|Z_i|^3\right) < \infty$. Let $s_n^2 = \sum_{i=1}^n E\left(Z_i^2\right)$ and $L_n = s_n^{-3} \cdot \sum_{i=1}^n E\left(|Z_i|^3\right)$. If $F_n(x)$ denotes the distribution function defined as*

$$F_n(x) = P\left(\frac{\sum_{i=1}^n Z_i}{s_n} \leq x\right)$$

then, for any $x \in \mathbb{R}$, we have

$$\left|F_n(x) - N(x)\right| \leq \frac{A \cdot L_n}{(1+|x|)^3}$$

where $N(\cdot)$ is the standard normal CDF *and A is an absolute positive constant.*

By making use of Theorem 7.1, we can provide a bound on the difference between the conditional variance call price $C(\sigma_K) = E\left[\left(RV_n - \sigma_K^2\right)_+ \middle| \mathcal{F}_T^W\right]$ and the conditional normal approximation $C^n(\sigma_K)$ in (9).

Proposition 7.1. *If $\sigma_K^2 \leq M_n$, then*

$$\left|C(\sigma_K) - C^n(\sigma_K)\right| \leq \frac{A \cdot \Sigma_n \cdot L_n}{2\left(1 + \frac{M_n - \sigma_K^2}{\Sigma_n}\right)^2}.$$

If $\sigma_K^2 > M_n$, then

$$\left|C(\sigma_K) - C^n(\sigma_K)\right| \leq A \cdot \Sigma_n \cdot L_n \cdot \left(1 - \frac{1}{2\left(1 + \frac{\sigma_K^2 - M_n}{\Sigma_n}\right)^2}\right).$$

where $L_n = s_n^{-3} \cdot \sum_{i=1}^n E\left(|Y_{n,i}|^3\right)$ with $Y_{n,i}$ as follows:

$$Y_{n,i} = \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \left(\mu_{n,i}^2 + \sigma_{n,i}^2\right) \tag{29}$$

□

Proof of Proposition 7.1 Integration by parts shows that for any random variable X with $E|X| < \infty$ and distribution function $F(x)$ we have

$$\mathbb{E}\left(X - \sigma_K^2\right)_+ = \mathbb{E}(X) - \sigma_K^2 + \int_{-\infty}^{\sigma_K^2} F(x)dx. \tag{30}$$

For a fixed $n \geq 1$, we consider the sequence of independent random variables $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}$ defined in (29). We have $\mathbb{E}\left(Y_{n,i}\right) = 0$ and $\mathbb{E}\left(|Y_{n,i}|^3\right) < \infty$; the exact formula for $\mathbb{E}\left(|Y_{n,i}|^3\right)$ can be found in Lemma 7.1. Using that

$$\frac{\sum_{i=1}^n Y_{n,i}}{s_n} = \frac{1}{\Sigma_n}\left(\mathrm{RV}_n - \frac{\sum_{i=0}^n \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta}\right)$$

the Theorem 7.1 of Bikelis implies

$$\left|F_{RV_n}\left(\Sigma_n \cdot x + M_n\right) - N(x)\right| \leq \frac{A \cdot L_n}{(1 + |x|)^3}$$

or, equivalently

$$\left|F_{RV_n}(x) - N\left(\frac{x - M_n}{\Sigma_n}\right)\right| \leq \frac{A \cdot L_n}{\left(1 + \left|\frac{x - M_n}{\Sigma_n}\right|\right)^3} \tag{31}$$

where F_{RV_n} denotes the conditional distribution of the discretely sampled variance and $L_n = s_n^{-3} \cdot \sum_{i=1}^n \mathbb{E}\left(|Y_{n,i}|^3\right)$. By (30) and (31) we have

$$\left|C(\sigma_K) - C^n(\sigma_K)\right| \leq \int_{-\infty}^{\sigma_K^2} \left|F_{RV_n}(x) - N\left(\frac{x - M_n}{\Sigma_n}\right)\right| dx = \int_{-\infty}^{\sigma_K^2} \frac{A \cdot L_n}{\left(1 + \left|\frac{x - M_n}{\Sigma_n}\right|\right)^3} dx.$$

Making the change of variable $\frac{x - M_n}{\Sigma_n} = y$ the remaining calculations are straightforward. For example, in the case $\sigma_K^2 \leq M_n$ the integral becomes

$$\int_{-\infty}^{\frac{\sigma_K^2 - M_n}{\Sigma_n}} \frac{A \cdot \Sigma_n \cdot L_n}{(1 - y)^3} dy = \frac{A \cdot \Sigma_n \cdot L_n}{2\left(1 + \frac{M_n - \sigma_K^2}{\Sigma_n}\right)^2}.$$

The case $\sigma_K^2 > M_n$ is solved similarly. □

Proof of Lemma 2.1 In the main text, we showed that, in the Black-Scholes model, $RV_n \stackrel{d}{=} \frac{\sigma^2}{n} \cdot \chi'(n, \lambda)$ where

$$\lambda = \frac{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 T}{\sigma^2}$$

and $\chi^{'}(n,\lambda)$ denotes the non-central chi-square distribution with n degrees of freedom and non-centrality parameter λ. We recall the density of a $\chi^{'}(n,\lambda)$ random variable:

$$f_{\chi'}(x;\lambda,n) = \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \cdot \left(\frac{\lambda}{2}\right)^i}{i!} \cdot f_{\chi}(x;n+2i)$$

where $f_{\chi}(x;n)$ denotes the PDF of a chi-square random variable with n degrees of freedom :

$$f_{\chi}(x;n) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} \cdot 1_{x>0}.$$

It is straightforward to show that $x \cdot f_{\chi}(x;n) = n \cdot f_{\chi}(x;n+2)$, which in turn allows us to write

$$\begin{aligned} x \cdot f_{\chi'}(x;\lambda,n) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \cdot \left(\frac{\lambda}{2}\right)^i}{i!} \cdot (n+2i) \cdot f_{\chi}(x;n+2+2i) \\ &= n \cdot f_{\chi'}(x;\lambda,n+2) + \lambda \cdot f_{\chi'}(x;\lambda,n+4). \end{aligned} \tag{32}$$

The expectation to compute becomes

$$\mathbb{E}\left(\frac{\sigma^2}{n} \cdot \chi^{'}(n,\lambda) - \sigma_K^2\right)_+ = \int_{\frac{\sigma_K^2 \cdot n}{\sigma^2}}^{\infty} \left(\frac{\sigma^2}{n} \cdot x - \sigma_K^2\right) \cdot f_{\chi'}(x;\lambda,n)\, dx$$

Using property (32) derived above, the result follows immediately. □

Lemma 7.1. *If we let*

$$Y_{n,i} = \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \left(\mu_{n,i}^2 + \sigma_{n,i}^2\right)$$

we have

$$\begin{aligned} \mathbb{E}\left|Y_{n,i}\right|^3 &= \sigma_{n,i}^6 \cdot \Big(\Psi(d_+) \cdot \phi(d_+) - \Psi(d_-) \cdot \phi(d_-) + (48\alpha^2+16) \cdot (N(d_-) - N(d_+)) + \\ &\quad 24\alpha^2 + 9\Big) \end{aligned}$$

where

$$d_{\pm} = \frac{-\mu_{n,i} \pm \sqrt{\mu_{n,i}^2 + \sigma_{n,i}^2}}{\sigma_{n,i}}.$$

and

$$\Psi(d) = 2d^5 + 12\alpha d^4 + (24\alpha^2+4)d^3 + (16\alpha^3+24\alpha)d^2 + (48\alpha^2+18)d + 32\alpha^3 + 60\alpha$$

with $\alpha = \frac{\mu_{n,i}}{\sigma_{n,i}}$.

Proof To simplify notation we drop the subscripts and let $\mu_{n,i} = \mu$ and $\sigma_{n,i} = \sigma$. We note that

$$Y_{n,i} \stackrel{d}{=} (\mu + \sigma \cdot N(0,1))^2 - \mu^2 - \sigma^2$$

where $N(0,1)$ denotes a standard normal variable. We thus have to compute

$$\int_{-\infty}^{\infty} \left|(\mu + \sigma \cdot x)^2 - \mu^2 - \sigma^2\right|^3 \cdot \phi(x)dx$$

where $\phi(x)$ denotes the standard normal density. By solving

$$\begin{aligned} (\mu + \sigma \cdot x)^2 - \mu^2 - \sigma^2 &= 0 \Leftrightarrow \\ \sigma^2 \cdot x^2 + 2\mu\sigma \cdot x - \sigma^2 = 0 \end{aligned}$$

we obtain the roots

$$d_{\pm} = \frac{-\mu \pm \sqrt{\mu^2 + \sigma^2}}{\sigma}.$$

Separating the positive and negative regions of the modulus, the integration becomes

$$\sigma^6 \cdot \left[\int_{-\infty}^{d_-} \left(x^2 + 2\alpha \cdot x - 1\right)^3 \cdot \phi(x)dx - \int_{d_-}^{d_+} \left(x^2 + 2\alpha \cdot x - 1\right)^3 \cdot \phi(x)dx \right.$$
$$\left. + \int_{d_+}^{\infty} \left(x^2 + 2\alpha \cdot x - 1\right)^3 \cdot \phi(x)dx \right]$$

where we put $\alpha = \frac{\mu}{\sigma}$. To compute these integrals, we note that by letting $u_n(x) = \int x^n \cdot \phi(x)dx$ and making use of the relationship

$$u_n(x) = -x^{n-1} \cdot \phi(x) + (n-1) \cdot u_{n-2}(x)$$

with $u_0(x) = N(x)$ and $u_1(x) = -\phi(x)$, we obtain

$$\begin{aligned} u_2(x) &= -x\phi(x) + N(x) \\ u_3(x) &= -(x^2 + 2) \cdot \phi(x) \\ u_4(x) &= -(x^3 + 3x) \cdot \phi(x) + 3 \cdot N(x) \\ u_5(x) &= -(x^4 + 4x^2 + 8) \cdot \phi(x) \\ u_6(x) &= -(x^5 + 5x^3 + 15x) \cdot \phi(x) + 15 \cdot N(x). \end{aligned}$$

Finally, the remaining steps involve only algebraic calculations to arrive at the result stated in the lemma. □

8 Appendix C

Table 2: OTM option prices under parameter sets $\mathcal{P}_3$ and $\mathcal{P}_4$. "OTM put" refers to an option with moneyness (Strike/v_0) 0.6, "ATM" to an option with moneyness 1 and "OTM call" to an option with moneyness 1.5. Prices are in percentages. The computational time is in seconds.

	T = 1 month				T = 6 months				T = 1 year			
	OTM put	ATM	OTM call	Comp. time	OTM put	ATM	OTM call	Comp. time	OTM put	ATM	OTM call	Comp. time
1 - Daily sampling frequency												
Parameter set $\mathcal{P}_3$												
Monte Carlo	0.0749	0.4327	0.1939	19.8938	0.1600	0.5515	0.5727	96.6546	0.1432	0.5191	0.7949	195.4242
Cont. sampling	0.0416	0.3322	0.0911	0.0161	0.1521	0.5368	0.5451	0.0458	0.1385	0.5110	0.7850	0.0530
Sepp's approx.	0.1112	0.4314	0.1475	0.0251	0.1691	0.5495	0.5529	0.1067	0.1498	0.5183	0.7896	0.1552
Cond. Normal	0.0902	0.4357	0.1791	32.6495	0.1586	0.5486	0.5573	111.2686	0.1424	0.5182	0.7926	218.1198
Simp. Cond. Normal	0.0854	0.4306	0.1751	0.4930	0.1566	0.5457	0.5572	0.5934	0.1406	0.5163	0.7861	0.6044
Simp. Cond. BS	0.0771	0.4277	0.1788	47.3588	0.1574	0.5462	0.5500	52.0469	0.1414	0.5145	0.7806	62.4184
Discr. Adj.	0.0706	0.4341	0.1903	0.0096	0.1564	0.5452	0.5570	0.0123	0.1407	0.5149	0.7905	0.0321
Parameter set $\mathcal{P}_4$												
Monte Carlo	0.0687	0.4330	0.1997	19.7739	0.1592	0.5512	0.5699	99.7854	0.1434	0.5200	0.7955	193.9963
Cont. sampling	0.0416	0.3322	0.0911	0.0159	0.1521	0.5368	0.5451	0.0171	0.1385	0.5110	0.7850	0.0076
Sepp's approx.	0.1112	0.4314	0.1475	0.0273	0.1691	0.5495	0.5529	0.1030	0.1498	0.5183	0.7896	0.1581
Cond. Normal	0.0902	0.4360	0.1792	32.8768	0.1591	0.5490	0.5653	111.9654	0.1421	0.5168	0.7895	213.4786
Simp. Cond. Normal	0.0851	0.4304	0.1748	0.5587	0.1548	0.5439	0.5549	0.5861	0.1397	0.5119	0.7952	0.6010
Simp. Cond. BS	0.0763	0.4259	0.1765	46.9367	0.1590	0.5494	0.5585	53.1714	0.1420	0.5174	0.7962	62.9918
Discr. Adj.	0.0706	0.4341	0.1903	0.0098	0.1564	0.5452	0.5570	0.0639	0.1407	0.5149	0.7905	0.0071
2 - Sampling frequency every five days												
Parameter set $\mathcal{P}_3$												
Monte Carlo	0.2215	0.6839	0.4535	18.8741	0.1839	0.5971	0.6282	101.5694	0.1594	0.5483	0.8345	190.0032
Cont. sampling	0.0416	0.3322	0.0911	0.0116	0.1521	0.5368	0.5451	0.0099	0.1385	0.5110	0.7850	0.0076
Sepp's approx.	0.2927	0.6934	0.3877	0.0113	0.2290	0.6003	0.5865	0.0280	0.1892	0.5482	0.8081	0.0467
Cond. Normal	0.3375	0.7360	0.4232	24.7000	0.1908	0.5978	0.6179	43.1629	0.1594	0.5435	0.8241	73.5120
Simp. Cond. Normal	0.3184	0.7199	0.4174	0.5207	0.1802	0.5883	0.6016	0.6477	0.1511	0.5339	0.8057	0.5877
Simp. Cond. BS	0.2251	0.6778	0.4330	33.9967	0.1769	0.5856	0.5996	47.9848	0.1496	0.5312	0.7980	52.7286
Discr. Adj.	0.1938	0.8669	0.6121	0.0101	0.1736	0.5790	0.6052	0.0092	0.1495	0.5311	0.8125	0.0480
Parameter set $\mathcal{P}_4$												
Monte Carlo	0.2085	0.6819	0.4665	18.9837	0.1848	0.6048	0.6519	96.6289	0.1608	0.5533	0.8445	193.8747
Cont. sampling	0.0416	0.3322	0.0911	0.0261	0.1521	0.5368	0.5451	0.0077	0.1385	0.5110	0.7850	0.0110
Sepp's approx.	0.2927	0.6934	0.3877	0.0105	0.2290	0.6003	0.5865	0.0263	0.1892	0.5482	0.8081	0.0493
Cond. Normal	0.3374	0.7360	0.4227	24.0948	0.1902	0.5972	0.6173	45.0983	0.1599	0.5443	0.8251	71.7491
Simp. Cond. Normal	0.3188	0.7210	0.4174	0.5726	0.1789	0.5845	0.6080	0.4564	0.1492	0.5286	0.8127	0.4923
Simp. Cond. BS	0.2259	0.6803	0.4281	34.3939	0.1724	0.5743	0.6032	46.6547	0.1479	0.5300	0.8157	53.3185
Discr. Adj.	0.1938	0.8669	0.6121	0.0096	0.1736	0.5790	0.6052	0.0097	0.1495	0.5311	0.8125	0.0841

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